Hilbert's 1990 ICM lecture. The 23 problems.

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In August1900 at the occasion of of the second International Congres of Mathematicians, in Paris that year, David Hilbert, then all of 38 years young, gave his lecture on Mathematical Problems. That lecture and even more the written version of it has been, or so it would seem[1], of great influence on the development of mathematics in the 20-th century. Partly because of the stature of the lecturer which was still to grow considerable in the decades to come, partly because the problems were well chosen [2], partly because they breathed a coherent view of what mathematics is all about, and perhaps most of all because of the incurable optimism in it all, a flat denial of Dubois-Reymond's "Ignoramus et ignoramibus".[3]

The published version, see [48, 49], contains 23 problems. Of these Hilbert discussed only 10 in the lecture itself, viz numbers 1, 2, 6, 7, 8, 13, 16, 19, 21, 22. The 23 problems, together with short, mainly bibliographical comments, are briefly listed below using the short title descriptions from [49].

Three general references are [1] (all 23 problems), [14] (all problems except 1, 3, 16), [56] (all problems except 4, 9, 14; special emphasis on developments from 1975-1992).

Two semipopular accounts of the problems, their solutions or solution attempts, and the people who worked on them are [38, 115]. The account below is mostly based on [46], and the references quoted there.

Problem 1. Cantor's problem on the cardinal number of the continuum.

More colloquially also known as the *continuum hypothesis*. It can be stated as

"Every uncountable subset of the real numbers, $\mathbf{R}$, has the same cardinality as $\mathbf{R}$, or as $2^{\aleph_0} = \aleph_1$".

Solved by K Gödel, [36] and P J Cohen, [18] in the (unexpected) sense that the continuum hypothesis is independent of ZFC, the *Zermelo-Frankel axioms* of set theory complete with the axiom of choice. This means that one can add the continuum hypothesis to ZFC without introducing inconsistencies (that were not already present) (Gödel); one can also add the negation of the continuum hypothesis (Cohen) without introducing inconsistencies. Gödel and Cohen also showed that the axiom of choice is independent of ZF.

Perhaps even more important than the solution of the problem itself are the techniques of Cohen forcing and Boolean valued models that resulted. These have 'uncountably' many applications by now.

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[1] There is no way to be absolutely sure of course. One can hardly run the 20-th century again to see how things would have gone without that lecture.

[2] Difficult, certainly, but not seemingly impossibly so, and often quite attractive. Hilbert worked hard for a year or so on his lecture. It is not really difficult to pose a list of nearly impossible problems but that is not what Hilbert did.

[3] We do not know and we shall not know (are not supposed to know).

Problem 2. The compatibility of the arithmetical axioms.

Solved (in a negative sense) by K Gödel, [35]. This is the socalled Gödel incompleteness theorem. It roughly says that in every system that is strong enough to do a reasonable amount of arithmetic there are statements that are not provable within that system and whose negation is also not provable. For a popular account, see [74]. Positive results (using techniques that Hilbert would not have allowed) are due to G Gentzen (1936) and P S Novikov (1941), see [1, 14].

Problem 3, The equality of the volumes of two tetrahedra of equal bases and equal altitudes.

More precisely the problem was to show that two such polyhedra can be transformed into each other by cutting and pasting (as is the case for triangles, the analogous problem in dimension 2). Whence the name 'scissors congruence problems'.

Solved in the negative sense by Hilbert's student Max Dehn in 1900, [25] (actually before Hilbert's lecture was delivered) and (at least partially) by R. Bricard in 1896, [13]. As it turned out, there is besides the volume one more quantity that remains invariant under cutting and pasting, the *Dehn invariant*. In higher dimensions the same problem can be studied and there are the *Hadwiger invariants*. In dimension 3 the Dehn invariant is the only extra invariant besides volume, i.e. tetrahedra with the same Dehn invariant and the same volume are scissors congruent, Sydler (1965), see [95].

Problem 4. Problem of the straight line as the shortest distance between two points.

This problem asks for the construction of all metrics in which the usual lines of projective space (or pieces of them) are geodesics. The first work on this was by Hilbert's student G Hamel, [42]. In particular he pointed out that the problem needed to be precisized and one should ask for all *Desarguesian* spaces in which straight lines are the shortest distances between points[4]. Nowadays, the problem is considered (basically) solved in the form of the following (generalized) Pogorelov theorem, [84, 85, 105]:

Any n-dimensional Desarguesian space of class C^{n+2}, $n \geq 2$, can be obtained by the BB construction.

Here the BB construction is an integral geometry based technique for obtaining Desarguesian spaces due to Blaschke, [12], and Busemann, [15]. The differentiability class restriction is necessary; otherwise there are Desarguesian spaces that do not come from the BB construction, see [105], which is also recommended as a very good survey on the 4-th problem.

Problem 5. Lie's concept of a continuous group of transformations without the assumption of the differentiability of the functions defining the group.

Solved in 1952 by Gleason and Montgomery and Zippin, [34, 70], in the form of the theorem "Every locally Euclidean topological group is a Lie group and even a real analytic group". For a much simplified (but nonstandard) treatment see [50]. The cases of compact topological groups and commutative topological groups were handled earlier by Pontryagin (1934) and von Neumann (1933).

This is perhaps the only one of Hilbert's problems that gave not rise to a host of subsequent investigations and problems and concepts. This happens but rarely. True, as Davis writes in his discussion of the first problem in [14], after Gödel's work there was some 20 years of stagnation in set theory. But Gödel's work did set people thinking about computability, recursiveness and the like, a most important development that prepared things for modern computer science and vast new parts of logic.

Problem 6. Mathematical treatment of the axioms of physics.

Very far from solved in any way, though there are (many bits and pieces of) axiom systems that have been investigated in depth. See [113] for an extensive discussion of Hilbert's own ideas, von Neumann's work and much more. There are for instance the *Wightman axioms* (also called *Gärding-Wightman axioms*) and the *Osterwalder-Schrader axioms* of quantum field

[4] Otherwise there are two many solutions. 'Desarguesian', of course, refers to the wel known Desargues theorem in plane projective geometry.

theory. There is von Neumann's axiomatization of quantum mechanics, following work of Nordheim and Hilbert himself. More recently there is the definition of topological field theories and conformal field theories, sources of very fruitful interactions between mathematics and physics, [63, 96, 99, 100, 109, 114]. Note that these are not really axiomatizations from the ground up (like Euclidean gerometry) but are more aptly termed relative axiomatizations in that they take an existing body of knowledge (like, say, differential topology) as given.

Quite early in the 20-th century, in direct response to Hilbert's questions, there were Carathéodory's axiomatizations of thermodynamics (1909) and of special relativity (1924), Hamel's axiomatization of mechanics (1903); and another independent axiomatization of special relativity by Robb (1914). And finally there was von Mises' axiomatization of probability (a field specifically mentioned by Hilbert in his elucidation of problem 6) followed by the definitive axiomatization by Kolmogorov, [60].

Preliminary to the axiomatization of quantum mechanics there was the development of Hilbert space, operators, infinite matrices, eigenvalues, integral equations, ... According to [89], p 183, Hilbert remarks that he developed this theory on purely mathematical grounds and even called it spectral analysis without any idea that it would later be much related to the real spectra of physics.

Problem 7. Irrationality and transcendence of certain numbers.

The numbers in question are of the form α^{β} with α algebraic and β algebraic and irrational. For instance $2^{\sqrt{2}}$ amd $e^{\pi}=i^{-2i}$. Solved in 1934 by A.O.Gel'fond and Th.Schneider (the *Gel'fond-Schneider theorem*). For the general method, the *Gel'fond-Baker method*, see e.g. [108]. A large part of [31] is devoted to Hilbert's seventh problem and related questions.

It is interesting to note that in a lecture of Hilbert in 1919 he remarked that he was optimistic to see the Riemann hypothesis solved in his lifetime, that perhaps the youngest member in the audience would see the solution of the Fermat problem, but that no one in the audience woud see the transcendence of $2^{\sqrt{2}}$.

Problem 8. Problems of prime numbers.

This one is usually known as the *Riemann hypothesis* and is the most famous and important of the unsolved conjectures in mathematics. The Riemann zeta-function of the complex variable s is given for $\mathrm{Re}(s)>1$ by $\varsigma(s)=\sum_{i=1}^{\infty}n^{-s}$ and it has an analytic continuation to the whole s-plane to a meromorphic function with one simple pole at $s=-1$ with residue 1. There are zeros for $s=-2,-4,\cdots$. These are referred to as trivial zeros. The Riemann hypothesis now says that all other zeros are of the form $s=\frac{1}{2}+i\tau$. It is known that the first 1.5 billion zeroes (arranged by increasing positive imaginary parts) are simple and lie on the critical line $\mathrm{Re}(s)=\frac{1}{2}$, [110]; it is also known that more than 40% of the zeros satisfy the Riemann hypothesis, [19, 64, 101].

The zeta function in algebraic geometry, $\varsigma_X(s)$, is a meromorphic function of a complex variable s that describes the arithmetic of algebraic varieties, X, over finite fields or of schemes of finite type over the integers. If X is $Spec(\mathbf{Z})$ one recovers the Riemann zeta function and if X is of finite type over $Spec(\mathbf{Z})$ there result the Dedeking zeta functions for the corresponding number fields.

A. Weil formulated a number of far ranging conjectures concerning zeta functions of varieties over finite fields (and proved them for curves). After the necessary cohomological tools for this were developed by A Grothendieck, M Artin and J-L Verdier these conjectures were proved by P Deligne, [27]. See [80] for much more detail.

Problem 9. Proof of the most general law of reciprocity in any number field.

Consider the question of whether an integer a is a quadratic residue modulo a prime number p or not, where a is not divisible by p. I.e. the question is whether a can be written in the form $a=b^2+kp$ for some integers b and k or not. In the first case write $\left(\frac{a}{p}\right)=1$, in the second $\left(\frac{a}{p}\right)=-1$. This is the Legendre symbol. The Gauss reciprocity theorem now says that for two different odd prime numbers $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{\frac{p-1}{2}\frac{q-1}{2}}$.

E Artin in 1927 gave reciprocity laws for general number fields; [6]. A great generalization of the Gauss reciprocity law had already been established by Hilbert himself in 1895, 1896. See [104] for more details and also for information as to how the question of reciprocity laws lead to (Abelian) class field theory, the subject of problem 12, see below. The analogous question of reciprocity laws for function fields was settled by Shafarevich in 1948, the *Shafarevich reciprocity law*, see [102].

Problem 10. Determination of the solvability of a Diophantine equation.

A Diophantine equation in a finite number of variables is an equation $P(x_1,\cdots,x_n)=0$ where P is a polynomial over the integers. It is solvable if there are integral solutions. For instance the Fermat equation $x^n+y^n=z^n$ for a given natural number n which has infinitely many solutions for n=1,2 and no solutions for $n\geq 3$. The problem asks for a finite sequence of tests (that can be applied to any such equation) to determine whether a Diophantine equation has solutions or not.

The solution is negative: there is no such algorithm, Matiyasevich (1979). This is a fairly immediate consequence of the main theorem in the field:

Every listable set of natural numbers is Diophantine.

See [23] for a description of the various concepts (though the meaning is intuitively rather clear).

One consequence of the main theorem is that there exists an integral polynomial such that the positive values of this polynomial on the natural numbers are precisely the prime numbers, [87], a result that made many mathematicians doubt that the main theorem, at that time still a conjecture, could possibly be true.

For a discussion of various refinements and extensions of problem 10, see [83].

Problem 11. Quadratic forms with any algebraic numerical coefficients.

This asks for the classification of quadratic forms over algebraic number fields. More precisely, a quadratic form over a (number) field K is an expression of the form $q=\sum_{i\leq j}q_{i,j}x_ix_j$ in the variables $x_1,\cdots,x_n$ with coefficients in K. Two such forms $q,\ q'$ are equivalent if there is an invertible linear substitution $x_i'=\sum t_{i,j}x_j$ such that $q'(x_1',\cdots,x_n')=q(x_1,\cdots,x_n)$. The problem is to classify quadratic forms up to this equivalence. This is solved by the *Hasse-Minkovsky theorem* and the *Hasse invariant*, [44]. The Hasse-Minkovsky theorem says that two quadratic forms over a number field K are equivalent if and only if they are equivalent over all of the local fields $K_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ of K. For instance for $K=\mathbf{Q}$, the rational numbers, two forms over $\mathbf{Q}$ are equivalent if and only if they are equivalent over the extensions $\mathbf{R}$, the real numbers, and the p-adic numbers $\mathbf{Q}_p$ for all prime numbers p[5]. This reduces the probelm to classification over local fields. And that is handled by the Hasse invariant (besides rank and discriminant). It is interesting to note that the definition of the Hasse invariant uses the Hilbert symbol and thus links to reciprocity (problem 9). See [68] and [76] for a great deal more informaton on the theory of quadratic forms.

Problem 12. Extension of Kronecker's theorem on Abelian fields to any algebraic realm of rationality.

The Kronecker-Weber theorem says that the maximal Abelian (meaning Abelian Galois group) extension $\mathbf{Q}^{ab}$ of the rational numbers is obtained by adjoining to $\mathbf{Q}$ all the roots of unity. This has two parts: on the one hand it gives an explicit contruction of $\mathbf{Q}^{ab}$; on the other it calculates the Galois group $\mathrm{Gal}(\mathbf{Q}^{ab}/\mathbf{Q})$. The second part has been nicely generalized for any number field (and also more generally). This is the topic of class field theory, which started with

[5] The real numbers are the completion of the rational numbers for the standard absolute value; the p-adic numbers are the completion for the non-Archimedean norm $\| a/b \|=2^{-v_p(a)+v_p(b)}$, which makes a rational number a/b, $(a,b)=1$, small if the numerator is divisible by a large power of the prime p.

Takagi, [106]. Since then the subject has gone through 6 or 7 incarnations (revolutions). See [7, 17, 45, 75, 112] for some of these.

The first half fared less well (explicit generation) except for the 'complex multiplication' case and local fields, [67]. But see also [51].

Nowadays there is great interest in and great progress on 'non-Abelian class field theory' in the form of the conjectured Langlands correspondence. In the local case (now proved for GL_n) this is a correspondence between representations of degree r of $\mathrm{Gal}(K^{sep} / K)$[6], or rather a dense subgroup W_K of it called the Weil group[7] and certain representations of $\mathrm{GL}_r(K)$. For the global case $\mathrm{GL}_r(K)$ is replaced by $GL_r(\mathbf{A})$ where $\mathbf{A}$ is the ring of adèles of K. The correspondence is also supposed to satisfy a number of strong extra properties. In case $r=1$ Abelian class field theory is recovered. No less than 4 invited lectures at the latest ICM, [65] were about the Langlands corespondence. Also there have been five Séminaire Bourbaki reports on the matter in recent years, giving another indication of how important the matter is considered to be, [16, 32, 43, 62].

Problem 13. Impossibility of the solution of the general equation of the 7-th degree by means of functions of only two variables.

This problem is nowadays seen as a mixture of two parts: a specific algebraic (or analytic) one concerning equations of degree 7, which remains unsolved, and a '*superposition problem*': can every continuous function in n variables be written as a superposition of continuous functions of two variables. The latter problem was solved by V I Arnol'd and A N Kolmogorov in 1956, 1957, [3, 61] : each continuous function of n variables can be written as a composite (superposition) of continous functions of two variables.
A composite function is one obtained by substituting other functions for the variables in the first functions. So, as an example, $f(x,y,z) = F(g(x,y), h(z, k(y,z)))$ is a function of three variables that is a composite of functions of two variables. Thus, for instance, all rational functions in any number of variables, can be obtained as composites of $x+y$, $x-y$, xy, x/y, The picture changes drastically if differentiability or analyticity conditions are imposed. For instance, there are analytic functions of n variables that cannot be written as composites of analytic functions of fewer variables.

The reason that the two parts of the problem occur together is that by Tschirnhausen transformations the general equation of degree 7 can be reduced to something of the form $X^7 + xX^3 + yX^2 + zX + 1 = 0$ (but no further) and the solutions of this equation as functions of x, y, z were considered to be candidates for functions of three variables that cannot be written as composites of functions of two variables.

Problem 14. Proof of the finiteness of certain complete systems of functions.

The precise form of the problem is as follows: Let K be a field in between a field k and the field of rational functions $k(x_1, x_2, \cdots, x_n)$ in n variables over k: $k \subset K \subset k(x_1, \cdots, x_n)$. Is it true that $K \cap k[x_1, \cdots, x_n]$ is finitely generated over k. The motivation came from positive answers (by Hilbert for instance) in a number of important cases where there is a group, G, acting on k^n and K is the field of G-invariant rational functions. A counterexample, precisely in this setting of rings of invariants, was given by M Nagata in 1959, [73]. However in the invariants case finite generation is true if the group is reductive; this is for instance the case if G is semisimple and k is of characteristic zero, [72].

Problem 15. Rigorous foundation of Schubert's enumerative calculus.

The problem is to justify and precisize H Schubert's '*principle of conservation of numbers*' under suitable continuous deformations. Mostly intersection numbers. For instance to prove rigorously that there are indeed, see [97], 666 841 048 quadric surfaces tangent to 9 given quadric surfaces in space. There are a great number of such principles of conservation of

[6] K^{sep} is the separable closure of K; if K is of charactieristic zero this is the algebraic closure.

[7] Not to be confused with the Weyl group of a simple Lie algebra.

numbers in intersection theory, [21], and cohomology and differential topology. Indeed one version of another of this idea is often the basis of definitions in singular cases.

In spite of a great deal of progress, see loc. cit., there remains much to be done to obtain a true *enumerative geometry* such as Schubert dreamt of.

And in fact more is required than just a good intersection theory that takes care of multiplicities. One also needs to give the collection of, say, all quadric surfaces in space the structure of something like an algebraic variety, i.e. something to which intersection theory can be applied. This is a fundamental subfield of algebraic geometry, starting with the question, which goes back to Riemann, as to on how many parameters a given kind of structure depends (how many moduli are needed in the phraseology of the 19-th century, which explains the terminology 'moduli space' in algebraic geometry).

Problem 16. Problem of the topology of algebraic curves and surfaces.

Even in its original formulation, this problem splits into two parts.

First, the topology of real algebraic varieties. For instance an algebraic real curve in the projective plane splits up in a number of ovals (topological cicles) and the question is which configurations are possible. For degree 6 this was finally solved by D A Gudkov (1970), see [40] for this and more. There are severe constraints on the configurations that are possible. Early important work on this is due to V Ragsdale, [88]. However, her conjectures have been fairly recently disproved by Itenberg and Viro, [55].

The second part concerns the topology of limit cycles of dynamical systems. A first problem here is the *Dulac conjecture* on the finiteness of the number of limit cycles of vector fields in the plane. For polynomial vecor fields this was settled in the positive sense by Yu S Il'yashenko (1970). See [4, 52, 53, 94] for this and much more.

Problem 17. Expression of definite forms by squares.

The problem is the following. Consider a rational function of n variables over the reals which in all points where it is defined takes nonnegative values. Does it follow that it can be written as a sum of squares (of rational functions). This was solved by E Artin in 1927, [5]. To solve the problem Artin invented the theory of formally real fields,. which has meanwhile other applications as well. For a definite function on a real irreducible algebraic variety of dimension d the *Pfister theorem* says that no more than 2^d terms are needed to express it as a sum of squares, [82].

Problem 18. Building up of space from congruent polyhedra.

This problem has three parts (in its original formulation).

(18a). Show that there are only finitely many types of subgroups of the group $E(n)$ of isometries of $\mathbf{R}^n$ with compact fundamental domain. Solved in 1910 by L Bieberbach, [11]. The subgroups in question are now called *Bieberbach groups.*

(18b). Tiling of space by a single polyhedron which is not a fundamental domain as in (18a). More generally also nonperiodic tilings of space. A *monohedral tiling* is a tiling in which all tiles are congruent to one fixed set T. If moreover the tiling is not one that comes from a fundamental domain of a group of motions one speaks of an *anisohedral tiling*. In one sense Hilbert's problem 18b was settled by K Reinhardt (1928), [90], who found an anisohedral tiling in $\mathbf{R}^3$, and H Heesch (1935) who found a non-convex anisohedral polygon in the plane that admits a periodic monohedral tiling, [47]. The tile of Heesch was actually produced as a roof tile and these tiles form the covering of the Göttingen Rathaus.

There also exists convex anisohedral pentagons, [57].

On the other hand, this circle of problems is still is a very lively topic today, see [98] for a recent survey.

For instance, the convex polytopes that can give a monohedral tiling of $\mathbf{R}^d$ have not yet been classified, even for the plane.

One important theory that emerged is that of the Penrose tilings and quasi-crystals, see [24]. As another example of one of the problems that emerged, it is unknown at this time (2003) which polyominos tile the whole plane, [37]. A *polyomino* is a connected figure abtained by taking n identical unit squares and connecting them along common edges.

(18c) Densest packing of spheres. Still unsolved in general. The densest packing of circles in the plane is the familiar hexagonal one, A Thue (1910, completed by L Fejes Tóth in 1940), [30, 107]. Conjecturally the densest packing in three space is the lattice packing A_3, the face centred cubic. This is the Kepler conjecture of 1610. This packing is indeed the densest lattice packing (Gauss), but conceivably there could be denser nonlattice packings (as can happen in certain higher dimensions). In 1998 T C Hales and S P Ferguson announced a proof of the Kepler conjecture. However, only two of the eight papers involved have sofar been published, both in 1997. The announced proof relies heavily on the computer checking of some 5000 special cases, a not dissimilar situation as 30 years ago with regard to the four colour conjecture. Still there are grounds that the proof will turn out to be substantially correct, [78].

The Leech lattice is conjecturally the densest packing in 24 dimensions. The densest lattice packings in dimensions 1-8 are known. In dimensions 10, 11, 13 there are packings that are denser than any lattice packing. See the standard reference [20].

Problem 19. Are the solutions of the regular problems in the calculus of variations always necessarily analytic.

This problem links to the 20-th problem through the Euler-Lagrange equation of the variational calculus. The variational problems meant are of the form: find a function $u{:}\overline{\Omega} \to \mathbf{R}$ tht is of class $C^1(\Omega) \cap C^0(\Omega)$ and is such that amoung all function of this class the integral $I[u] = \int_\Omega F(x, u(x), p(x))dx$ is minimal and satisfies a Dirichlet type boundary condition $u(x) = \varphi(x)$ for $x \in \partial\Omega$. Here Ω is a bounded open set in $\mathbf{R}^n$, $\overline{\Omega}$ is its closure, $\partial\Omega$ is its boundary, and $p(x) = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$. The function F is given and satisfies the regularity (and convexity) conditions $F \in C^2$ and $\left(\frac{\partial^2 F}{\partial p_i \partial p_j}\right) > 0$.

The correponding Euer Lagrange equation is

$$\sum_{i,j=1}^n F_{p_i p_j}(x,u,p)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n (F_{p_i u} p_i + F_{p_i x_i}) = F_u.$$

Positive results on the analyticity for nonlinear elliptic partial equations were first obtained by S N Bernshtejn in 1903 and in more or less definite form by I G Petrovskii (1937), [10, 81].

Problem 20. The general problem of boundary values.

In 1900, the general matter of boundary value problems and generalized solutions to differential equations, as Hilbert wisely specified, was in its very beginning. The amount of work accomplished since is enormous in achievement and volume and includes generalized solution ideas (weak solutions) such as the *distributions* of Dirac, Sobolev and Schwartz (see [111]) and, rather recently for the nonlinear case, *generalized function algebras*, [77, 92, 93] to get , among other, around the problem that distributions do not have a good multiplication.

Problem 21. Proof of the existence of linear differential equations having a perscribed monodromy group.

Consider a system of n first order linear differential equations $y'(z) = A(z)y$ on the Riemann sphere $\mathbf{P}^1$ where A(z) is meromorphic. Let Σ be the set of poles of $A(z)$. Such a system has an n-dimensional space S of soltions. Following a solution along a loop around one of the poles by analytic continuation gives a possibly different solution. This gives a representation of the fundamental group $\pi_1(\mathbf{P}^1 \setminus \Sigma) \to GL_n(\mathbf{C})$, the monodromy representation of the system of differential equations.

The question is now whether every representation of the fundamental group comes from a system of differential equations where it is moreover required that all the poles of $A(z)$ are simple. For a long time it was thought that this was true by the work of L Plemelj, G Birkhoff, I

Lappo-Danilevskij. But, in 1989 A. Bolibrukh found counterexamples. However, if extra apparent singularities are allowed: singularities where the monodromy is trivial, there is a positive solution, [2, 8].

As formulated by Hilbert the 21-st problem had to do with an n-th order linear differential equation of Fuchsian type $y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$, which means that a_i has at most a pole of order i. Here the answer is again negative if no apparent singularities are allowed and positive if this is allowed.

In the modern literature the question is studied in the form of connections on a bundle over any Riemann surface or even in far more general situations, [26, 91].

Problem 22. Uniformization of analytic relations by means of automorphic functions.

This is the uniformization problem, i.e representing (most of) an algebraic or analytic manifold parametrically by single-valued functions.

For instance $(\sin(t), \cos(t))$ and $\left(\frac{2u}{u^2+1}, \frac{u^2-1}{u^2+1}\right)$ t and u complex variables, both parametrize the Riemann surface of $z^2 + w^2 = 1$.The (complex) dimension one case was solved by H Poincaré and P Koebe in 1907 in the form of the *Koebe general uniformization theorem* that a Riemann surface topologically equivalent to a domain in the extended complex plane is also conformally equivalent to such a domain, and the *Poincaré-Koebe theorem* or *Klein-Poincaré uniformization theorem*, see [41]. For higher (complex) dimension things are still largely open and that also holds for a variety of generalizations, loc. cit. and [1, 14].

Problem 23. Further development of the methods of the calculus of variations.

As in problem 19 the problem is to find curves, surfaces, ... that minimize certain integrals. Many problems is physics are formulated in terms of variational principles. Hilbert felt that the calculus of variations had been somewhat neglected and had a number of precise ideas of how to go further. Though there were already in 1900 a great many results in the calculus of variations, [59], very much more has been developed since both as regards what may be termed the classical calculus of variations, [33, 69], and numerous more modern offshoots such as: optimal control, [66, 86] and dynamic programming, [9]; the calculus of variations in the large started by Marston Morse, [71]; the theory of minimal differential geometric objects (geodesics, minimal surfaces, Plateau problem), [22, 39, 79, 116]]; variational inequalities, [58]; links with convex analysis, [29]. Treating variational problems as optimization problems in infinite dimensional (function) spaces brings a unifying perspective, [54].

As is only natural the idea of having another new stimulating list of problems for the 21-first century has arisen. There was such an attempt in 1974 at the occasion of the review of the then current status of the Hilbert problems, and there are 27 groups of problems in the proceedings of that meeting [14]. They do not seem to have been all that successful as a guide to research. More recently, Stephen Smale formulated a list, [103]. Still more recently, the seven millenium problems were formulated by the new Clay Institute of Mathematics, see [28] for a popular account and go to <http://www.claymath.org> for the offcial descriptions of these seven problems[8]. These seven are far more deeply imbedded in technically sophisticated mathematics (except one). They are:

The Riemann hypothesis
Yang-Mills theory and the mass gap hypothesis
The P vs NP problem
The Navier-Stokes equations
The Poincaré conjecture
The Birch and Swinnerton-Dyer conjecture
The Hodge conjecture

Each of them carries prize money of 1 million dollars. It remains to be seen whether they will do as much as is hoped to attract brilliant young people to research mathematics.

[8] Some of these descriptions are extraordinarily well written.

Perhaps not. For much of the 20-th century there may have been a sort of general pervasive feeling that there is something like a vast, potentially complete, unique (rigid) edifice comstituting mathematics. And perhaps that accounts for the feelings of (foundational) anxiety that one senses when reading accounts of the progress of mathematics on the Hilbert problems.

Today seems to be less a period of problem solving, nor a period of large theory building. Instead we seem to live in a period of discovery where new beautiful applications, interrelations and phenomena appear with astonishing frequency. It is a multiverse of many different axiom systems, of different models of even something as basic as the real numbers, of infinitely many different differentially structures on the space-time, that we live in[9]. It is a world of many different chunks of mathematics, not necessarily provably compatible, at least until we come up with new ideas of what it means to be provable. Nor need all of mathematics to be compatible. Meanwhile mathematicians go happily about the delightful business of discovering (or inventing) and describing new beauty and insights.

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